3/2/26

Return exams

A's 90–100
B's 80–89
C's 70–79
D 60–69
F <60

mean 95.2
median 96

exam 1 + solutions are posted on webpage

No office hours today (thesis defense)
No office hours Thurs (special seminar)
} except by appt

Recall:

<u>Def</u> Let W be a subset of $\mathbb{R}^n$. Then W is a (linear) <u>subspace</u> if

(a) W contains $\vec{0}$ (the 0-vector in $\mathbb{R}^n$)

(b) W is closed under add'n of vectors

(c) W is closed under scalar mult.

<u>Ex</u> In $\mathbb{R}^2$ let W be the line $2x+3y=0$. Is W a subspace of $\mathbb{R}^2$??

Yes! Say (a,b) and (c,d) are in W and $k \in \mathbb{R}$

We know
$$\left.\begin{array}{l} 2a+3b=0 \\ 2c+3d=0 \end{array}\right\} \Rightarrow 2(a+c)+3(b+d)=0$$
$$\Rightarrow (a,b)+(c,d) = (a+c,\ b+d) \in W$$

and $k(2a+3b) = k(0) = 0$
$$\Rightarrow 2(ka)+3(kb)=0$$
$$\Rightarrow k(a,b) \in W$$

<u>Ex</u> In $\mathbb{R}^2$ let W be the line $2x+3y=1$. Is W a subspace of $\mathbb{R}^2$?

No: e.g. $(-1,1) \in W$ but $(-2,2) \notin W$.

Similarly, in $\mathbb{R}^3$ subspaces are either lines or planes thru $\vec{0}$

- planes thru $\vec{0}$
- lines thru $\vec{0}$
- $\vec{0}$ alone

Tie this in with the fact we've seen that kernels & images of lin. transf. are subspaces

Ex Consider the plane W defined by $2x + 3y + 4z = 0$ in $\mathbb{R}^3$.

(a) Find a lin transf $T: \mathbb{R}^3 \to \mathbb{R}^3$ whose kernel is W.
(b) Find a lin transf $T: \mathbb{R}^3 \to \mathbb{R}^3$ whose image is W.

(Either of these confirms that W is a subspace, thanks to what we've shown.)

(a) since $\begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2x + 3y + 4z$, W is the kernel of

$T: \mathbb{R}^3 \to \mathbb{R}^1$ defined by $\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$. But we need $\mathbb{R}^3 \to \mathbb{R}^3$.

For example we can use $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$.

check: $2x + 3y + 4z = 0$
$4x + 6y + 8z = 0$
$6x + 9y + 12z = 0$
$\rightsquigarrow \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (not quite RREF but we don't need that)

(b) Note $(-1, 2, -1) \in W$ and $(-3, 2, 0) \in W$ and $(-4, 4, -1) \in W$

So $\begin{bmatrix} -1 & -3 & -4 \\ 2 & 2 & 4 \\ -1 & 0 & -1 \end{bmatrix}$ defines $T: \mathbb{R}^3 \to \mathbb{R}^3$ whose image (= column space) is W.

Ex Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \end{bmatrix}$ defining $T: \mathbb{R}^3 \to \mathbb{R}^3$.

We know that image(T) is a <u>subspace</u> of $\mathbb{R}^3$, and that

$$\text{image}(T) = \text{col. space of } A = \left\{ \text{lin comb of } \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \right\}.$$

But note that $\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$ is <u>redundant</u> since from $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ we can get $\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$ as one of the lin. comb.

So image(T) is better described as $\left\{ \text{lin comb of } \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$.

Let's try to formalize this.

Def Let $\vec{v}_1, \ldots, \vec{v}_m$ be vectors in $\mathbb{R}^n$.

(a) Say $\vec{v}_i$ is one of the vectors in this list. Then $\vec{v}_i$ is <u>redundant</u> if $\vec{v}_i$ is a lin. comb of $\vec{v}_1, \ldots, \vec{v}_{i-1}$

(b) $\vec{v}_1, \ldots, \vec{v}_m$ are <u>linearly indep</u> if none of them is redundant.

Otherwise we say they are <u>linearly dependent</u>. We'll have a better way to check linear (in)dependence

(c) Let V be a subspace of $\mathbb{R}^n$ and $\vec{v}_1, \ldots, \vec{v}_m \in V$. We say $\vec{v}_1, \ldots, \vec{v}_m$ <u>form a basis of V</u> if they span V and are lin. indep.

<u>Remarks</u>

(a) it's a bit more common to ignore the order of the vectors and just ask if they're lin indep or lin dep. as we'll see in a sec. But we'll follow the book

(b) Suppose $\vec{v}_1, \ldots, \vec{v}_{i-1}$ are lin indep (so no redundancy) but $\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_i$ are not. Then $\vec{v}_i = a_1\vec{v}_1 + \cdots + a_{i-1}\vec{v}_{i-1}$ for some $a_1, \ldots, a_{i-1}$ not all 0. Thus $a_1\vec{v}_1 + \cdots + a_{i-1}\vec{v}_{i-1} + (-1)\vec{v}_i = 0$.

We'll come back to this.